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Angular momentum and separation constant in the Kerr metric

Fernando de Felice†

Theoretical Physics Institute, University of Alberta, Edmonton, Alberta, Canada T6G–2J1
and
Departamento de Física, Pontifícia Universidade Católica, CxP 38071, Rio de Janeiro, Brasil

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Abstract. The Kerr metric admits a Killing tensor which yields a second-order constant of motion for classical trajectories. We find an explicit expression for this constant using a particular coordinate system, the oblate spheroidal. Owing to the separability of the Hamilton–Jacobi equation in this system, it is easy to show that the second-order constant is, in fact, the square of the total angular momentum of the particle at infinity, corrected with terms which arise from the non-inertial character of the coordinate system.

1. Introduction

In the Kerr metric, the Hamilton–Jacobi equation is separable when use is made of the oblate spheroidal coordinate system (Carter 1968a, b).

Together with the particle rest energy and the two constants of motion which derive from the explicit symmetries of the Kerr space–time, namely the stationarity and the axisymmetry, the separability itself yields a fourth constant of motion. It is well known (Walker and Penrose 1970, Woodhouse 1975, Carter 1977) that a second-order constant of motion of this type is associated with a second rank symmetric tensor K_{ij} which satisfies an equation similar to the Killing equation, i.e.

$$K_{(ij;k)} = 0.$$

With this (Killing) tensor, the constant of motion can be expressed, in whatever coordinate system, as the invariant contraction of K_{ij} with the particle four-momentum P_i, i.e. $K^{ij}P_iP_j$. Although it has long been realised that this constant is somehow connected with the square of the particle total angular momentum to which it reduces in the limit of $a \to 0$, a being the rotational parameter of the Kerr metric, no explicit expression has been given so far in terms of the physical quantities which characterise the motion and the field. In this paper we find the solution to this problem using explicitly the oblate spheroidal coordinate system. In this system, in fact, the very separability of the Hamilton–Jacobi equation allows one to express the separation constant in terms of well-defined physical quantities.

Because we think that our result for the Kerr metric can be extended to more general stationary and axisymmetric solutions with separability properties, we first

† On leave of absence from Istituto di Fisica 'G Galilei', Università di Padova, 35100 Padova, Italy.

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write the equations of motion in the most general case (§ 2), then we specialise to the Kerr metric. In § 3 we describe the usefulness of working in terms of a Newtonian potential which is analogous to the Kerr metric in the sense that it gives the same description of the angular motion and of the asymptotic behaviour of the trajectories. We then derive the full expression of the total angular momentum of a particle in this Newtonian field, using oblate spheroidal coordinates, and finally in § 4 we interpret the role of the separation constant in the asymptotic expression of the angular momentum.

2. The equations of motion

The most general canonical form of a stationary and axisymmetric metric which yields a separable Hamilton–Jacobi equation is given by (here we use the same notation as in Carter (1968a))

$$\mathrm{d}s^2=\Sigma\left(\frac{\mathrm{d}r^2}{\Delta_r}+\frac{\mathrm{d}\mu^2}{\Delta_\mu}\right)+\Sigma^{-1}[\Delta_\mu(C_r\,\mathrm{d}t-Z_r\,\mathrm{d}\phi)^2-\Delta_r(C_\mu\,\mathrm{d}t-Z_\mu\,\mathrm{d}\phi)^2] \tag{1}$$

where $\Sigma=C_\mu Z_r-C_r Z_\mu$. Here C_μ and C_r are constants, while Z_μ, Z_r, Δ_μ and Δ_r are functions of the corresponding coordinate only (i.e. $Z_\mu=Z(\mu)$, etc). The coordinates r and μ are assumed to represent, respectively, a radial distance from the metric source and a latitudinal variation between a south and a north polar symmetry axis. The remaining coordinates ϕ and t, which are clearly ignorable in the metric (1), are related to the symmetries of the field and represent, respectively, an azimuth angle about the axis of symmetry and a time. The equations of motion for a free particle of mass m in the space–time (1) are easily derived from the Hamilton–Jacobi equation:

$$g^{ij}\frac{\partial s}{\partial x^i}\frac{\partial s}{\partial x^j}+m^2c^2=0 \tag{2}$$

where c is the velocity of light and s is the action integral over a Lagrangian of the form:

$$\mathscr{L}=\frac{1}{2}g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}. \tag{3}$$

Here, τ is an affine parameter along the trajectories.

In equation (2), g^{ij} is the inverse of the metric (1) and reads:

$$\left(\frac{\partial}{\partial s}\right)^2=\Sigma^{-1}\left[\Delta_\mu\left(\frac{\partial}{\partial\mu}\right)^2+\Delta_r\left(\frac{\partial}{\partial r}\right)^2+\frac{1}{\Delta_\mu}\left(C_\mu\frac{\partial}{\partial\phi}+Z_\mu\frac{\partial}{\partial t}\right)^2+\frac{1}{\Delta_r}\left(C_r\frac{\partial}{\partial\phi}+Z_r\frac{\partial}{\partial t}\right)^2\right]. \tag{4}$$

Hence equation (2) becomes:

$$\Delta_r P_r^2+\Delta_\mu P_\mu^2+\left(\frac{C_\mu^2}{\Delta_\mu}-\frac{C_r^2}{\Delta_r}\right)P_\phi^2+2\left(\frac{C_\mu Z_\mu}{\Delta_\mu}-\frac{C_r Z_r}{\Delta_r}\right)P_t P_\phi+\left(\frac{Z_\mu^2}{\Delta_\mu}-\frac{Z_r^2}{\Delta_r}\right)P_t^2+m^2c^2\Sigma=0 \tag{5}$$

with $P_i=\partial s/\partial x^i$. From the symmetries of the space–time (1) and the physical significance attached to the coordinates ϕ and t, we readily infer that P_ϕ and P_t are constants of the motion. In case the metric (1) is asymptotically flat, they are interpreted as the axial component of the angular momentum of the particle and (minus) its total energy as measured by a static observer at infinity.

We shall consider the latter case only; calling $P_\phi = l$ and $P_t = \epsilon/c$, equation (5) can be immediately separated as

$$\Delta_r P_r^2 - \frac{C_r}{\Delta_r} l^2 + \frac{2C_r Z_r}{\Delta_r}\frac{\epsilon}{c} l - \frac{Z_r^2}{\Delta_r}\frac{\epsilon^2}{c^2} + m^2c^2 C_\mu Z_r = -K \tag{6}$$

$$\Delta_\mu P_\mu^2 + \frac{C_\mu^2}{\Delta_\mu} l^2 - \frac{2C_\mu Z_\mu}{\Delta_\mu}\frac{\epsilon}{c} l + \frac{Z_\mu^2}{\Delta_\mu}\frac{\epsilon^2}{c^2} - m^2c^2 C_r Z_\mu = K \tag{7}$$

where K is the separation constant. Now the existence of a non-zero mixed component, $g_{\phi t}$, assures that metric (1) in general describes the gravitational field of a rotating system. The condition that this term vanishes is simply

$$C_r Z_r/\Delta_r = C_\mu Z_\mu/\Delta_\mu \tag{8}$$

which implies that both terms must be equal to some constant h. Using this condition in (1) we obtain the most general diagonal metric form with separability properties:

$$\mathrm{d}s^2 = \frac{\Sigma}{\Delta_r}\mathrm{d}r^2 + \frac{\Sigma}{\Delta_\mu}\mathrm{d}\mu^2 - \frac{C_\mu \Delta_r}{\Sigma_r}\mathrm{d}t^2 + \frac{\Delta_\mu Z_r^2}{\Sigma}\left(1 - \frac{C_r^2 \Delta_\mu}{C_\mu^2 \Delta_r}\right)\mathrm{d}\phi^2. \tag{9}$$

Spherical symmetry is attained if we put $h = 0$; with this choice, in order to preserve the right signature in the metric (1), we must have from (8):

$$C_r = Z_\mu = 0. \tag{10}$$

Using (10), equation (7) becomes

$$\Delta_\mu P_\mu^2 + \frac{1}{\Delta_\mu} l^2 = K \tag{11}$$

where we redefine l as $C_\mu l$; here, K is clearly the square of the total angular momentum of the particle. In the most general case, however, the total angular momentum is not conserved and its expression at an arbitrary point on the particle trajectory is rather difficult to obtain. If, however, we write equation (7) in the following way:

$$\Delta_\mu P_\mu^2 + \frac{l^2}{\Delta_\mu} = K + F(\mu) \tag{12}$$

where

$$F(\mu) = m^2c^2 C_r - \frac{Z_\mu^2}{\Delta_\mu}\frac{\epsilon^2}{c^2} + \frac{2Z_\mu}{\Delta_\mu}\frac{\epsilon}{c} l \tag{13}$$

and recall that we assumed the space–time to be asymptotically flat, then, with equation (12) being independent of r, we conjecture that the quantity $K + F(\mu)$, which is always positive, is the square of the total angular momentum of the particle at infinity.

In this case we would expect the asymptotic value of the angular momentum to be independent of the angular position of the particle. If $F(\mu)$ is not a constant, then the coordinate-dependent terms only arise from the peculiarities of the coordinate system we are using and therefore need to be carefully understood. We shall see how this is true for the Kerr metric.

3. The Kerr metric

The Kerr metric is a vacuum solution of Einstein's field equations (Kerr 1963) and a particular case of the metric (1). If we choose

$$\begin{array}{llll} \mu=\cos\theta & C_\mu=1 & C_r=a & \Delta_\mu=\sin^2\theta \\ \Delta_r=r^2-2Mr+a^2 & & Z_r=r^2+a^2 & Z_\mu=a\sin^2\theta \end{array} \tag{14}$$

we obtain the well-known solution in the standard Boyer and Lindquist (1967) form with oblate spheroidal coordinates:

$$ds^2=\Sigma\left(\frac{dr^2}{\Delta_r}+d\theta^2\right)+\frac{1}{\Sigma}\{\sin^2\theta[a\,dt-(r^2+a^2)\,d\theta]^2-\Delta_r(dt-a\sin^2\theta\,d\phi)^2\} \tag{15}$$

with $\Sigma=r^2+a^2\cos^2\theta$. Here M and a have the meaning of the mass and specific angular momentum of the metric source in units of length. From (14), equation (12) becomes:

$$P_\theta^2+\frac{l^2}{\sin^2\theta}=K+m^2c^2a^2\sin^2\theta+2a\frac{\epsilon}{c}l-\frac{\epsilon^2a^2}{c^2}\sin^2\theta. \tag{16}$$

Here the separation constant K is covariantly connected to a Killing tensor which, in the Kerr metric, reads (Walker and Penrose 1970):

$$\begin{aligned} K_{ij}\dot{x}^i\dot{x}^j &= \Sigma\left(\frac{a^2\sin^2\theta}{\Delta}\dot{r}^2+(r^2+a^2)\dot{\theta}^2\right) \\ &\quad+\Sigma^{-1}\{(r^2+a^2)\sin^2\theta[(r^2+a^2)\dot{\phi}-a\dot{t}]^2-\Delta a^2\sin^2\theta(a\sin^2\theta\dot{\phi}-\dot{t})^2\} \\ &= K+a^2m^2c^2 \end{aligned}$$

where $\dot{x}^i$ are the components of the particle four-velocity. Although K can take negative values, the quantity $C=K+a^2m^2c^2$ is always positive as can be easily seen by expressing (16) in terms of C.

Let us now define the new constant:

$$L=K-m^2c^2a^2\Gamma+2a(\epsilon l/c) \tag{17}$$

where $\Gamma=(\epsilon/mc^2)^2-1$; then equation (16) reduces to

$$P_\theta^2+l^2/\sin^2\theta=L+m^2c^2a^2\Gamma\cos^2\theta. \tag{18}$$

Before going further, let us specify the physical meaning of the constant Γ. From its definition we have

$$\Gamma=2E_k/mc^2 \tag{19}$$

where E_k is simply the kinetic energy of the particle at infinity. Using (19), equation (18) becomes

$$P_\theta^2+l^2/\sin^2\theta=L+2a^2E_k\cos^2\theta \tag{18}$$

where we have expressed all the quantities in units of the particle mass. Equation (18) is exactly the same angular equation one would obtain from the Hamilton–Jacobi equation in a Newtonian potential given by

$$V(r,\theta)=-\frac{GM_*r}{r^2+a^2\cos^2\theta} \tag{20}$$

where G is the gravitational constant and M_* is the mass of the field source in conventional units. Here r and θ are oblate spheroidal coordinates as in equation (15). In these coordinates it is easy to see that the Hamilton–Jacobi equation is separable, the separation constant being L. The reason why the angular equations are the same is simply that they are independent of G. Indeed, one would get the same equation were the particle moving in a flat space–time endowed with oblate spheroidal coordinates. As the separation constant L in (18) can be expressed in terms of asymptotic quantities only, there will be no loss of generality if we interpret L in the framework of Newtonian theory. The potential $V(r, \theta)$ is the Newtonian analogue to the Kerr metric first discussed by Keres (1967) and Israel (1970). A series expansion of this potential, i.e.

$$V(r,\theta) \simeq -\frac{GM_*}{r} + \frac{GM_* a^2}{r^3}\cos^2\theta + \dots \tag{21}$$

shows that the deviation from spherical symmetry is mainly due to a quadrupole moment which agrees with that of the Kerr metric. Despite the angular equations of motion being the same as those in the Kerr metric, the Newtonian radial equation differs from the relativistic one in that it does not contain the terms which describe the rotational dragging effects on local inertial frames.

These terms, however, do not affect the asymptotic limit as they tend to zero as $\sim al/r^3$. From (20) in fact the radial equation reads:

$$P_r^2 = (r^2+a^2)^{-1}\left(-L + 2E_k r^2 + 2GM_* r + \frac{a^2 l^2}{r^2+a^2}\right) \tag{22}$$

and in the limit of $r \to \infty$, $P_r^2 \to 2E_k$, as in the relativistic case (Carter 1968b)†. We shall now derive the explicit expression of the total angular momentum of a particle which moves in the potential (20).

In Newtonian mechanics the angular momentum is given by:

$$\boldsymbol{M} = 0\boldsymbol{Q} \times \boldsymbol{P} \tag{23}$$

where $0\boldsymbol{Q}$ is a radial vector to a point Q (figure 1), and $\boldsymbol{P}$ is the linear momentum vector of that point. After transforming to oblate spheroidal coordinates and forming the modulus ($M = |\boldsymbol{M}|$), one finds

$$M^2 = \left(\frac{a^2}{(r^2+a^2)^{1/2}}\sin\theta\cos\theta P^r - r(r^2+a^2)^{1/2}p^\theta\right)^2 + (r^2+a^2)\sin^2\theta(r^2+a^2\cos^2\theta)(p^\phi)^2. \tag{24}$$

From the metric (24) and the (covariant) components of the momentum given by equations (17) and (22), we finally have

$$M^2 = \frac{(r^2+a^2)}{(r^2+a^2\cos^2\theta)}\left[\frac{a^2\sin(2\theta)}{2(r^2+a^2)^{1/2}}\left(-L+2E_k r^2+2GM_* r+\frac{a^2l^2}{(r^2+a^2)}\right)^{1/2} - r\left(L+2E_k a^2\cos^2\theta - \frac{l^2}{\sin^2\theta}\right)^{1/2}\right]^2 + \frac{l^2(r^2+a^2\cos^2\theta)}{(r^2+a^2)\sin^2\theta}. \tag{25}$$

Although complicated, this expression becomes very simple in particular cases. For example, when $a = 0$, we have, as expected, $M^2 = L$; when $l = 0$, we deduce that the

† The same limit we would have had, did we consider the radial equation in flat space–time with the same coordinate system.

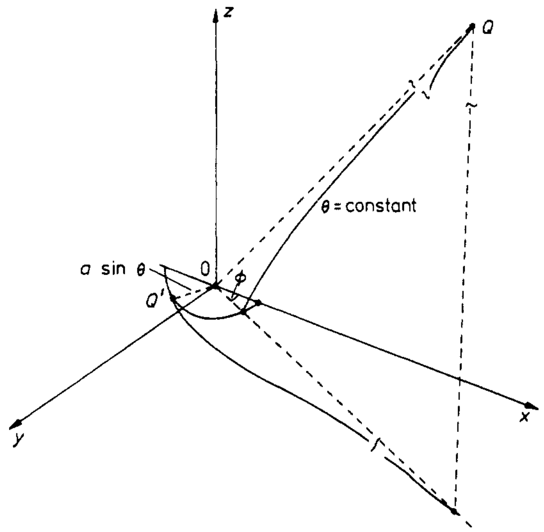

Figure 1.

motion on the axis $\theta = 0$ is stable (i.e. $M = 0$) if $L = -2E_k a^2$, a result which holds in the relativistic case too (Calvani and de Felice 1978). Finally, in the limit of $r \to \infty$ we have

$$M^2_\infty = L + 2E_k a^2 \cos^2 \theta \qquad (26)$$

in agreement with our conjecture at the end of § 2.

Here the θ-dependent term does not represent any action of the field as the potential (20) vanishes at infinity. However, although the space–time at infinity is flat, its geometrical description remains in terms of the oblate spheroidal coordinates. The term $2E_k a^2 \cos^2 \theta$ therefore simply describes a coordinate effect which we shall consider in the next section.

4. Interpretation of the separation constant

To understand the significance of the θ-dependent term in equation (26), one has to recall the property that oblate spheroidal coordinates associate to each point $Q \equiv \{r, \theta, \phi\}$ an 'origin' of the coordinate system which is located at the point $Q' \equiv \{x = -a \sin \theta \sin \phi, y = a \sin \theta \cos \phi, 0\}$ (see figure 1). In a spherically symmetric field for example, while with respect to Cartesian or spherical polar coordinates the centre of symmetry is the point $0 \equiv \{x = y = z = 0\}$ relative to any point in the space, in the oblate spheroidal coordinates each point Q 'recognises' as the centre of symmetry the corresponding point Q'. In the flat space–time, the total angular momentum is a constant quantity only if calculated with respect to the centre of symmetry relative to each particle's initial position. Obviously the point 0 is not the centre of symmetry for any of the points at infinity when we use oblate spheroidal coordinates except for those on the axis ($\theta = 0$). This is the reason why in (26) the angular momentum (which we recall is calculated relative to 0) is not a constant. What we expect to be a constant then is the angular momentum of the particle at a point Q, relative to the corresponding point

Q'. This is in fact the case. From classical mechanics we have that:

$$\underset{(Q')}{\boldsymbol{M}'} = \underset{(0)}{\boldsymbol{M}} - 0\boldsymbol{Q}' \times \boldsymbol{P} \tag{27}$$

where $0\boldsymbol{Q}'$ is a radial vector to the point Q' and $\boldsymbol{P}$ and $\underset{(0)}{\boldsymbol{M}}$ are the same as in (23).

Calculating the square modulus, we have, after some algebra,

$$\begin{aligned}\underset{(Q')}{M^2} = \underset{(0)}{M^2} + a^2 \sin^2\theta\left[(\cos\theta P^r - r\sin\theta P^\theta)^2 + \left(\frac{r\sin\theta P^r}{(r^2+a^2)^{1/2}} + (r^2+a^2)^{1/2}\cos\theta P^\theta\right)^2\right] \\ +2a\frac{lr\cos\theta}{(r^2+a^2)^{1/2}}(\cos\theta P^r - r\sin\theta P^\theta) \\ +2al\sin\theta\left(\frac{r\sin\theta P^r}{(r^2+a^2)^{1/2}} + (r^2+a^2)^{1/2}\cos\theta P^\theta\right).\end{aligned} \tag{28}$$

In the limit of $r \to \infty$ this becomes

$$\underset{(Q')}{M_\infty^2} = \underset{(0)}{M_\infty^2} + 2a^2 E_k \sin^2\theta + 2al\sqrt{2E_k} \tag{29}$$

and then from (26):

$$\underset{(Q')}{M_\infty^2} = L + 2a^2 E_k + 2al\sqrt{2E_k}. \tag{30}$$

This allows us to express the Newtonian separation constant L in terms of well-defined physical quantities and, therefore, from equation (17) we can also express the Kerr metric separation constant in terms of these, i.e.

$$K = \underset{(Q')}{M_\infty^2} - 2alc\left\{\left(\frac{\epsilon}{mc^2}\right) - \left[\left(\frac{\epsilon}{mc^2}\right)^2 - 1\right]^{1/2}\right\} \tag{31}$$

where K, $\underset{(Q')}{M_\infty^2}$ and l are in units of m.

In (31) the separation constant is expressed in terms of asymptotic quantities, and therefore it holds only if $\epsilon/mc^2 > 1$.

5. Conclusion

Despite the Kerr metric being asymptotically flat, the separation constant turns out to be not just the square of the particle total angular momentum at infinity, as one would expect, but that quantity corrected by terms proportional to the rotational parameter a. The reason is simply that at infinity the space–time is still described in terms of the oblate spheroidal coordinates with parameter a, and therefore the additional terms in (31) are fictitious contributions to the angular momentum by the non-inertial character of the coordinate system (see Note added in proof).

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Note added in proof. One can equivalently interpret these terms as the asymptotic contribution to the particle mechanical angular momentum by the angular momentum of the gravitational field itself. I am indebted to Professor J York for enlightening discussions on this point.

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